FINC 430 TA Session 7
Risk and Return
Solutions

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Formulas for return and risk

The expected return of a portfolio of two risky assets, *i* and *j*, is

$$E\left(r_{portfolio}\right) = xE(r_i) + (1-x)E\left(r_j\right)$$

$E(r_i)$ –Expected return of asset i
x - the percentage of wealth (called a portfolio weight) that is invested in asset i.

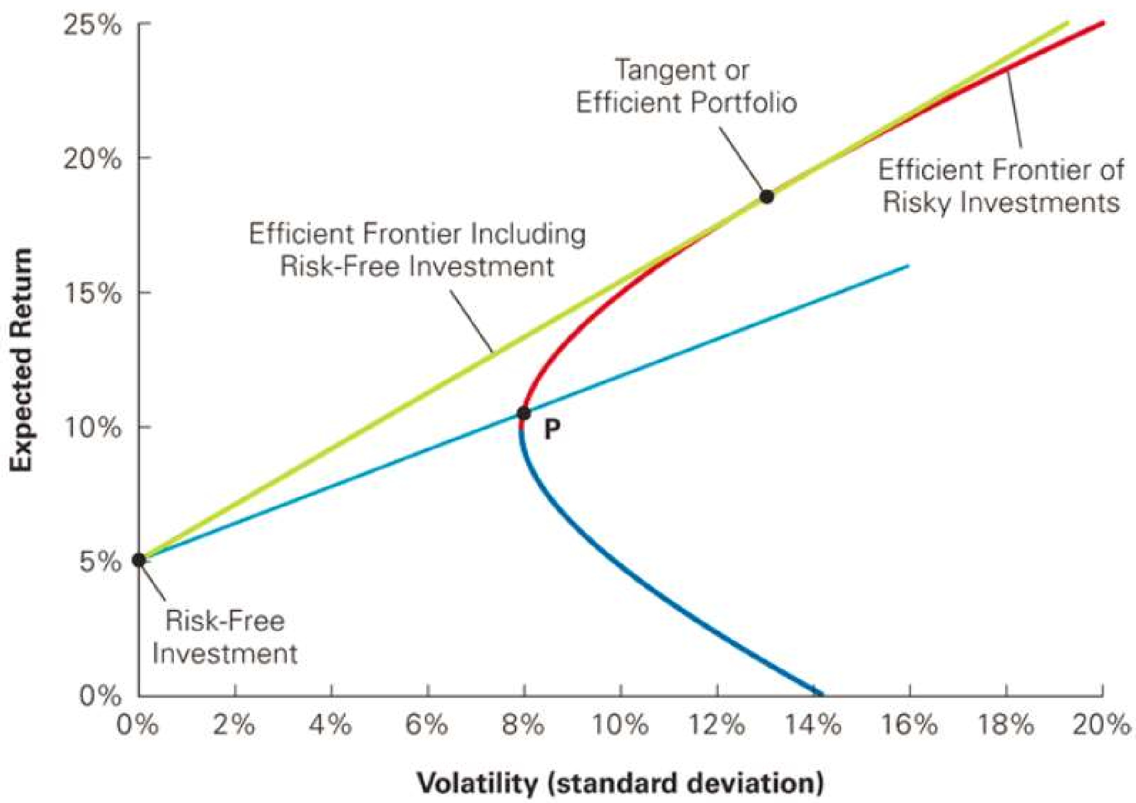
The risk (standard deviation) of a portfolio of two risky assets, i and j, is

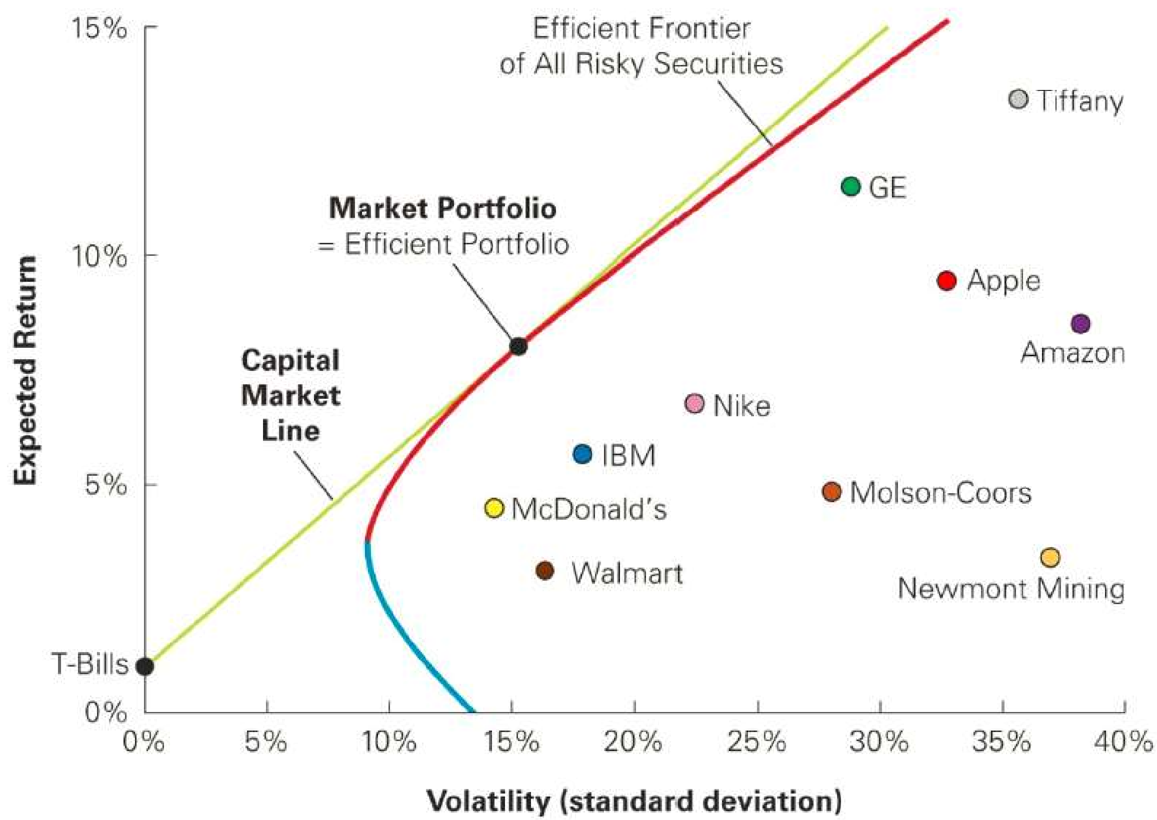
$$\mathrm{SD}\left(r_{portfolio}\right) = \left[x^2 VAR(r_i) + (1-x)^2 VAR\left(r_j\right) + 2x(1-x)Corr\left(r_i, r_j\right) \times SD(r_i) \times SD\left(r_j\right)\right]^{\frac{1}{2}}$$

VAR(r_i) –Variance of returns of asset i
SD(r_i) –Standard Deviation of returns of asset i
$Corr\left(r_i, r_j\right)$ is the correlation coefficient between securities i and j
Comes from formula for VAR(aX+bY)

Two fund separation

Case 1: There is a risk-free asset -> linear combination of market and risk-free
Case 2: There is not a risk-free asset-> combination of two frontier risky assets
Two fund theory – any two portfolios on mvf determine the entire mvf
Frontier (parabola) is traced out as portfolio mean is varied





For portfolios on the Capital Market Line (with risk free asset), we have

$$E\left(r_{portfolio}\right) = x_m E(r_m) + (1 - x_m)E\left(r_f\right)$$

$$\mathrm{SD}\left(r_{portfolio}\right) = x_m \mathrm{SD}(r_m) \text{ [Build up from formula]}$$

$E(r_m)$ is the expected return of the market portfolio
$\mathrm{SD}(r_m)$ is the standard deviation of returns of market portfolio
$E\left(r_f\right)$ is the expected return of the risk-free asset

Question 1

Which of the following statements are true?

A) A stock's return is perfectly positively correlated with itself

B) When the covariance equals 0, the stocks have no tendency to move either together or in opposition of one another

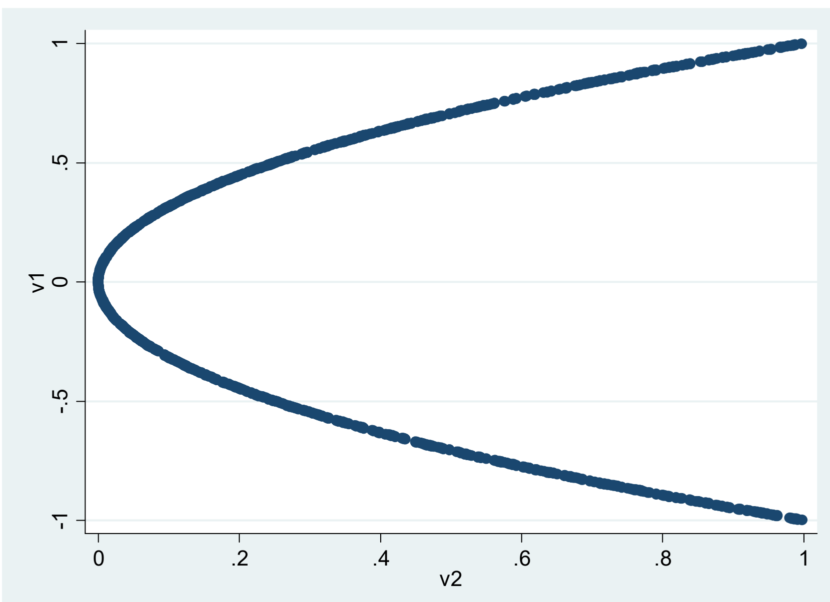
C) The closer the correlation is to -1, the more the returns tend to move in opposite directions

D) The variance of a portfolio depends only on the variance of the individual stocks

Answer: A, B, C

Not D because the variance of a portfolio depends on the variance and correlations of the individual stocks.

Caveat on B – correlation vs. independence



Here, I have a uniform [-1,1] random variable v1, and another variable v2 which is v1^2

Although they are uncorrelated [Statistical proof] they are obviously not independent!

In this class, however, we only care about linear relationships

Question 2

Which of the following statements are true?

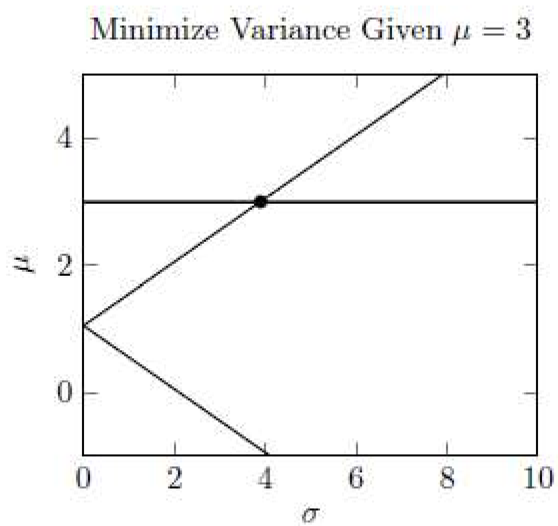
A) For a portfolio of assets with the same individual variances, when the correlation between assets is less than 1, the standard deviation of the portfolio's return is reduced due to diversification

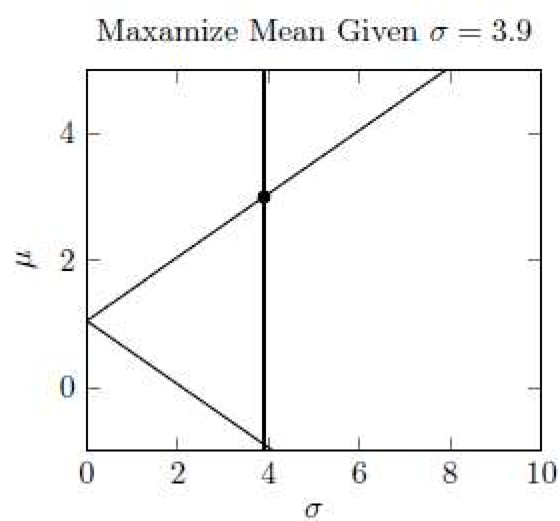
B) The efficient portfolios are those portfolios offering the highest possible expected return for a given level of volatility (i.e., portfolio return standard deviation)

C) The efficient portfolios are those portfolios offering the lowest possible volatility (i.e., portfolio return standard deviation) for a given level of expected return

D) The lower the correlation of the securities/assets in a portfolio, the lower the volatility we can obtain for the portfolio's return

Answer: A, B, C , D


Minimize Variance Given $\mu = 3$
4
2
0
μ
0
2
4
6
8
10
σ


Maxamize Mean Given $\sigma = 3.9$
4
2
0
μ
0
2
4
6
8
10
σ

Question 3

You have \$10,000 invested in a portfolio *p.* If you sell half and invest \$5,000 in an asset A whose return has a correlation of zero with the return on portfolio *p,* is it possible that your overall portfolio variance could go up?

Yes. The portfolio variance can go up if asset A has sufficiently high variance.

The new portfolio variance is

$$\begin{aligned}\sigma_{new}^2 &= x_p^2\sigma_p^2 + x_A^2\sigma_A^2 + 2x_p x_A \rho_{p,A}\sigma_p\sigma_A \\ &= 0.25\sigma_p^2 + 0.25\sigma_A^2 \\ &= 0.25 \times (\sigma_p^2 + \sigma_A^2)\end{aligned}$$

and $\sigma_{new}^2 > \sigma_p^2$ if σ_A^2 is sufficiently big (more than 3 times σ_p^2)

Question 4

True or False:

Suppose a mean-variance investor has access to two stocks only, and no riskless asset. Stock A has an expected return of 15 percent, stock B has an expected return of 10 percent. Both stocks have a standard deviation of 20 percent. Faced with this, an investor should never choose a positive portfolio share for stock B.

False, unless the assets have a positive correlation of one.

Investing a positive fraction in asset B will lower the expected portfolio return, but for correlation $\rho < 1$ will also lower the portfolio standard deviation. Depending on the investors degree of risk aversion, this may be preferred to investing everything in asset A. The lower ρ is, the higher the diversification benefit (for example, for a value of $\rho = -1$, a zero standard deviation for the portfolio is possible for appropriate portfolio shares).

Question 5

Your current portfolio consists of three assets, the common stock of Netscape and Wal-Mart combined with an investment in the riskless asset. You know the following about the stocks ($\rho_{i,j}$ denotes the correlation between asset i and asset j):

$$\rho_{Netscape,M} = 0.30 \qquad \rho_{WalMart,M} = 0.4$$
$$E(r_{Netscape}) = 0.148 \qquad E(r_{WalMart}) = 0.13$$
$$\sigma^2_{Netscape} = 0.64 \qquad \sigma^2_{WalMart} = 0.25$$

You also have the following information about the Market portfolio (M) and the riskless asset (f):

$$E(r_M) = 0.13 \qquad r_f = 0.04$$
$$\sigma^2_M = 0.04$$

Assume that individuals can borrow and lend at r_f and that all investors hold efficient portfolios (In other words, the market portfolio is the tangent portfolio). You have $200,000 invested in Netscape, $200,000 invested in Wal-Mart, and $100,000 invested in the riskless asset

(a) Assume that the correlation between Netscape and Wal-Mart, $\rho_{Netscape,WalMart}$, is 0.10. What is the variance of your portfolio?

(b) Find an efficient portfolio that has the same expected return as your current portfolio, but the lowest standard deviation possible. What is the standard deviation and variance of this portfolio?

(c) Find an efficient portfolio that has the same standard deviation as your current portfolio, but the highest expected return possible. What is the expected return of this portfolio?

(End of question)

(a) Assume that the correlation between Netscape and Wal-Mart, $\rho_{Netscape,WalMart}$, is 0.10. What is the variance of your portfolio?

We will use the formula for the variance of a portfolio of three assets and then simplify; since all the covariances and the variance term involving the riskfree asset are zero:

$$\begin{aligned}
\sigma_p^2 \\
&= x_{Netscape}^2 \sigma_{Netscape}^2 + x_{WalMart}^2 \sigma_{WalMart}^2 + x_f^2 \sigma_f^2 \\
&+ 2x_{Netscape} x_{WalMart} \times \rho_{Netscape,WalMart} \times \sigma_{Netscape} \times \sigma_{WalMart} \\
&+ 2x_{Netscape} x_f \times \sigma_{Netscape,f} + 2x_{WalMart} x_f \times \sigma_{WalMart,f} \\
&= 0.4^2 \times 0.64 + 0.4^2 \times 0.25 + 0 + 2 \times 0.4 \times 0.4 \times 0.10 \times 0.8 \times 0.5 + 0 + 0 \\
&= 0.1552
\end{aligned}$$

(The standard deviation is thus $\sigma_p = 39.395\%$.)

(b) Find an efficient portfolio that has the same expected return as your current portfolio, but the lowest standard deviation possible. What is the standard deviation and variance of this portfolio?

The expected return of the current portfolio is

$$E(r_p) = x_{Netscape}E(r_{Netscape}) + x_{WalMart}E(r_{WalMart}) + x_f E(r_f)$$
$$= 0.4(0.148) + 0.4(0.13) + 0.2(0.04)$$
$$= 0.1192$$

The efficient portfolio, call it *ep*1, has an expected return of 0.1192. Since the portfolio is made up of the market and the riskless asset, $E(r_{ep1}) = x_m E(r_m) + x_f E(r_f)$. Thus, $0.1192 = x_m \times 0.13 + (1 - x_m) \times 0.04$ and $x_m = 88\%$

Variance of the portfolio $\sigma_{ep1}^2 = x_m^2 \sigma_m^2 = 0.88^2 \times 0.04 = 0.0310$

Standard deviation of the portfolio $\sigma_{ep1} = 17.6\%$

(c) Find an efficient portfolio that has the same standard deviation as your current portfolio, but the highest expected return possible. What is the expected return of this portfolio?

This efficient portfolio, call it *ep2,* has a standard deviation of 39.395%. Since the portfolio is made up of the market and the riskless asset, $\sigma_{ep2} = x_m \times \sigma_m$. Thus, $39.395\% = x_m \times 20\%$ and $x_m = 197\%$ and so $x_f = -97\%$.

The expected return on this portfolio is

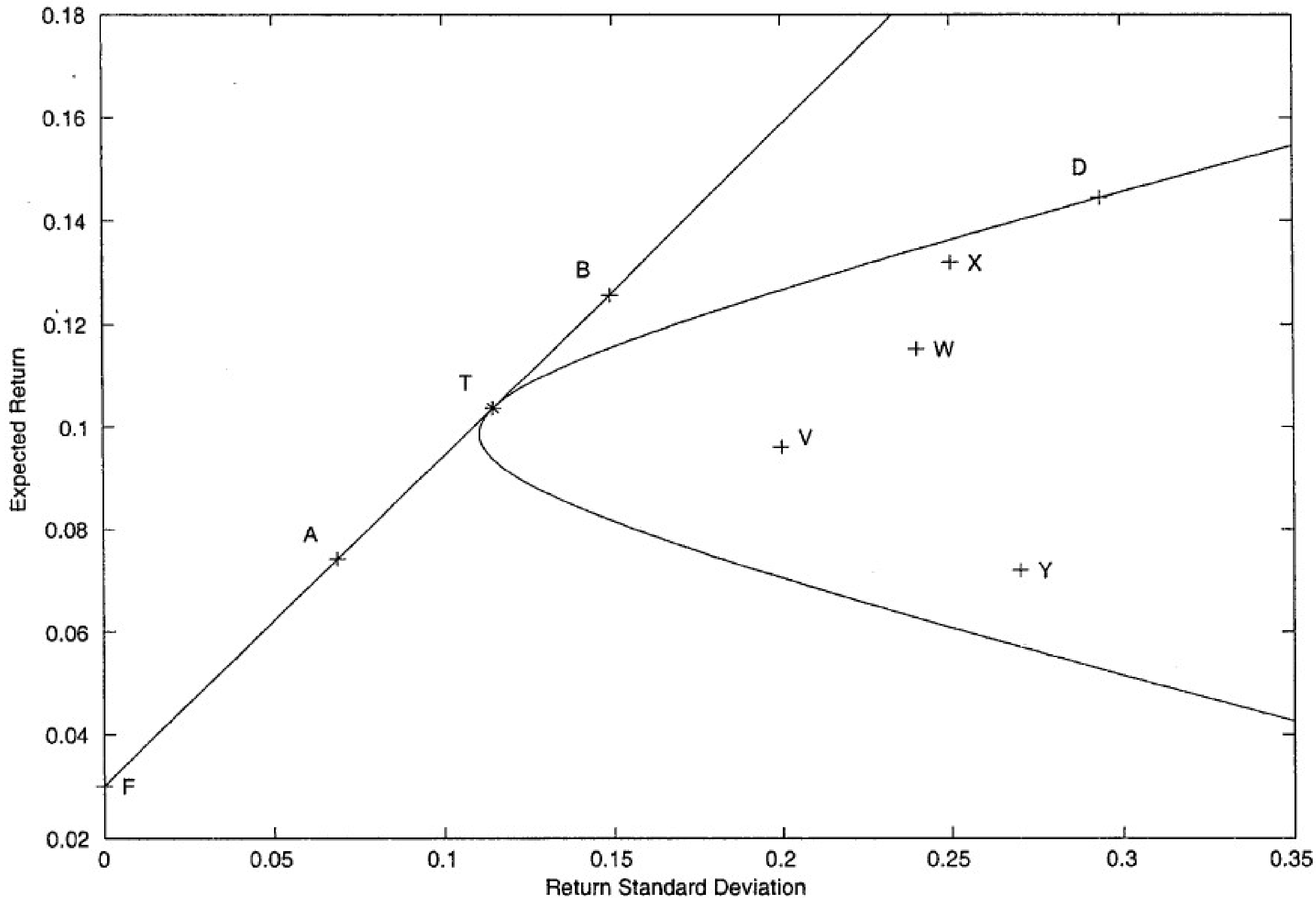
$$
\begin{aligned}
E\left(r_{ep1}\right) &= x_m E(r_m) + x_f E\left(r_f\right) \\
&= 1.97 \times 0.13 - 0.97 \times 0.04 \\
&= 21.73\%.
\end{aligned}
$$

Question 6

Suppose the *entire securities market* consists of a riskless asset, F, together with only 4 stocks: V, W, X, and Y. There are 200 Million shares of each of V and W outstanding, and 500 Million shares of each of X and Y outstanding. Assume that the price of stock X is equal to the price of stock Y.

The graph below illustrates the location of F and of the 4 stocks. Also shown is the minimum variance frontier for investing in V, W, X, and Y, the mean-variance efficient portfolio (denoted by T) of these 4 stocks, the capital allocation line through F and T, and two portfolios A and B on this capital allocation line. Furthermore, portfolio D is a portfolio of V, W, X, and Y.

Assume that portfolio T is the market portfolio of the 4 stocks (i.e. the market portfolio of risky assets).



Expected Return
Return Standard Deviation
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
B
D
X
W
T
V
A
Y
F

The following questions can be answered *without doing any math.*

(a) If you could invest in F and only one of the 4 stocks, which stock would you choose?

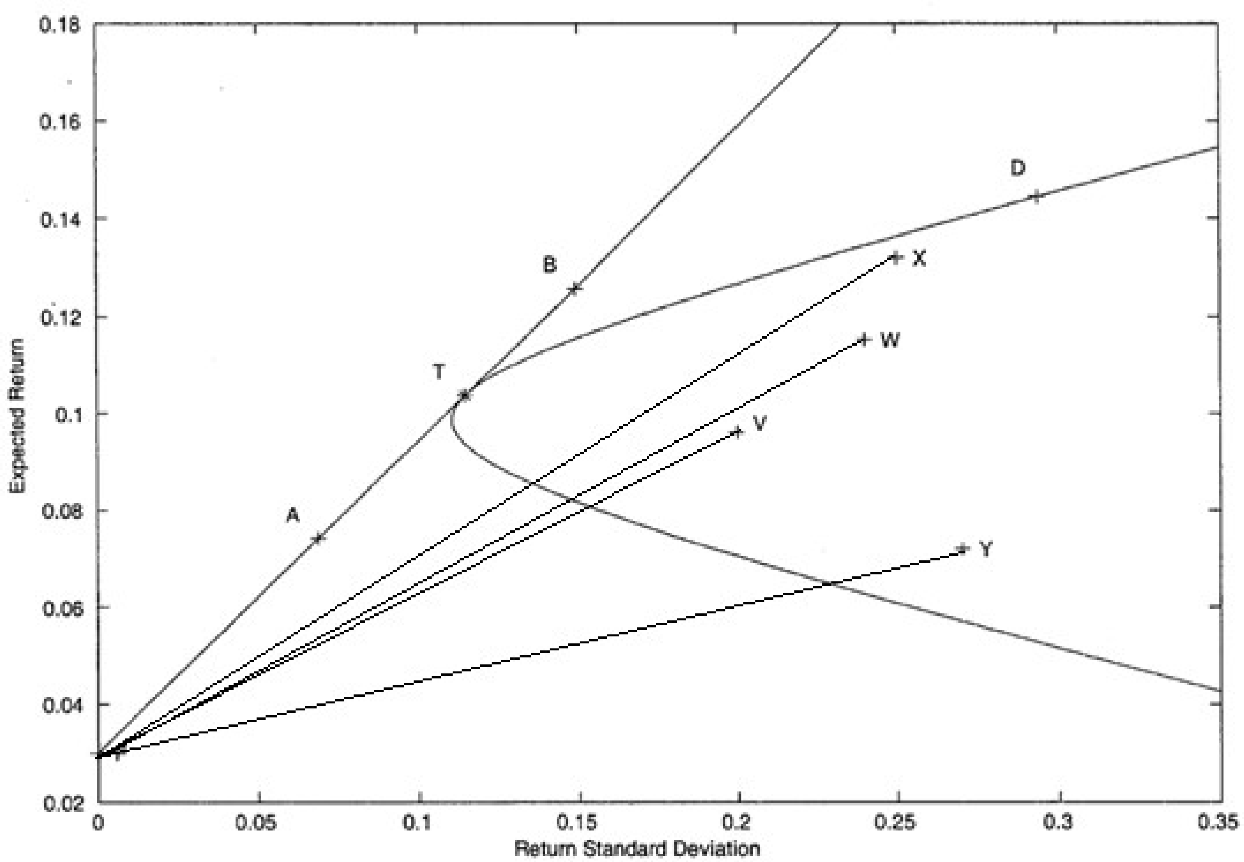
(b) Which of the portfolios A, B, and D can you be sure involve borrowing/short sales of some asset (i.e. a negative portfolio share for one or more assets)?

(c) If you hold an efficient portfolio and you own 300 shares of stock X, how many shares of stock Y do you own?

(End of question).

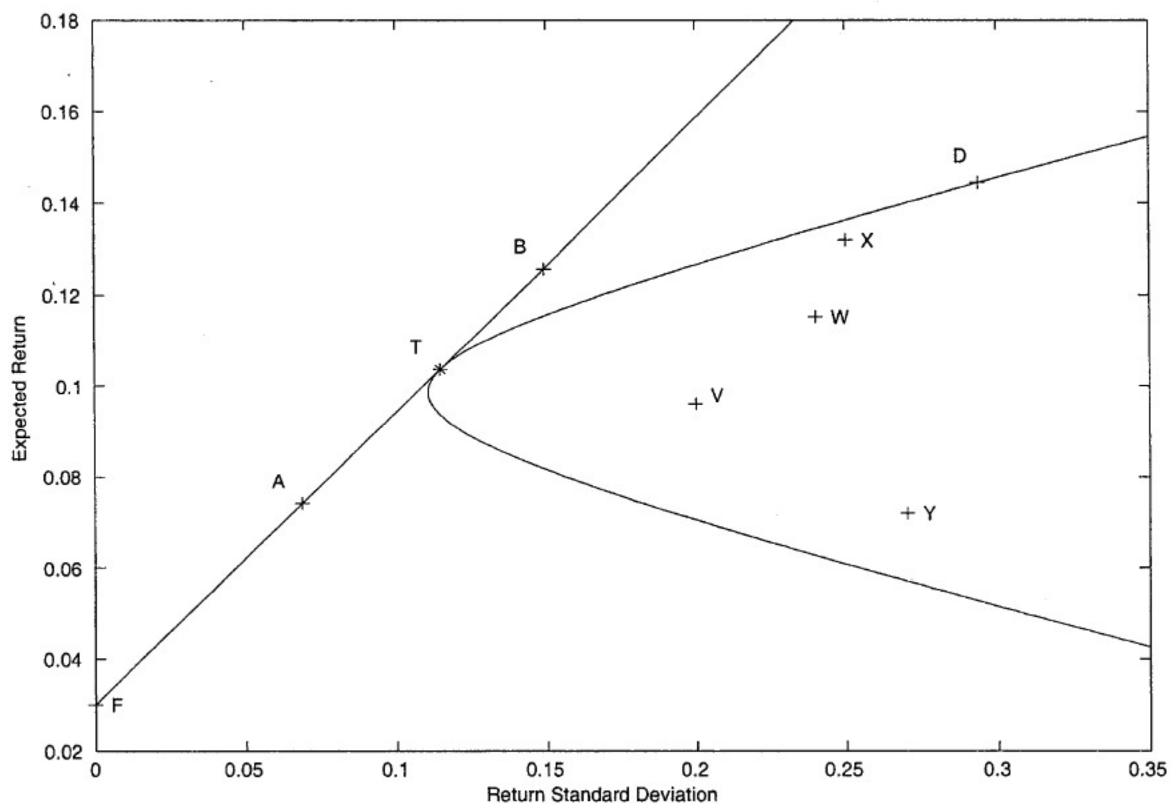
(a) If you could invest in F and only one of the 4 stocks, which stock would you choose?

X, since it would lead to the steepest capital allocation line. (In other words, the line joining X and F, which corresponds to portfolios of X and F, gives a higher expected return for any return standard deviation than the lines joining F with V, W or Y). See graph below with these lines added.



(b) Which of the portfolios A, B, and D can you be sure involve borrowing/short sales of some asset (i.e. a negative portfolio share for one or mote assets)?

B involves a negative portfolio share for the riskless asset F. D must involve a negative portfolio share for one or more of the 5 stocks since its expected return is higher than that of all 5 stocks



(c) If you hold an efficient portfolio and you own 300 shares of stock X, how many shares of stock Y do you own?

300 shares of stock Y.

Optimal portfolios are combinations of F and T, and T equals the market portfolio of stocks. Since the market value of X and Y is the same, and since the price of X and Y is the same, any investment in T must involve equal holdings of X and Y.

QUESTIONABLE QUESTIONS

Question 7

A New York Times article provided the following portfolio advice:

Investor Type	Percent of overall portfolio		
	Riskless Asset	Risky Bonds	Stocks
Conservative (C)	20	40	40
Moderate (M)	10	30	60
Aggressive (A)	0	20	80

(a) Is the advice provided consistent with two fund separation? Answer without doing any math.

The risky asset portfolios of the three investor types differ substantially:

Investor Type	Percent of risky asset portfolio	
	Risky Bonds	Stocks
Conservative (C)	50	50
Moderate (M)	33 1/3	66 2/3
Aggressive (A)	20	80

Suppose that expected returns are 12% for stocks, 7% for risky bonds, and 4% for the riskless asset. Standard deviations are 20% (i.e. 0.2) for stocks and 9% for risky bonds, and the correlation between the returns on stocks and risky bonds is 0.25.

(b) Work out the expected returns and standard deviations of the risky asset portfolios recommended for the three investor types (conservative (C), moderate (M), aggressive (A)). Call these three portfolios p^C, p^M, and p^A.

It turns out that neither of p^C, p^M, and p^A are the mean-variance efficient portfolio of risky bonds and stocks. The mean-variance efficient portfolio of risky assets, p^{MVE}, consists of investing 62% in risky bonds, and 38% in stocks (take my word for it, don't try to work it out). It has an expected return of 8.90% and a standard deviation of 10.49%.

Let us construct some better portfolio advice. For simplicity, focus on the aggressive investor.

(c) Construct a portfolio q of the riskless asset and the mean-variance efficient portfolio of stocks and risky bonds, p^{MVE}, which has the same standard deviation as p^A but the highest possible expected return. Do this by working out the portfolio share invested in the riskless asset and the portfolio share invested in the mean-variance efficient portfolio p^{MVE}, and work out the expected return of portfolio q.

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(a) Is the advice provided consistent with two fund separation? Answer without doing any math.

No. Two fund separation states that all investors should invest in a combination of the riskless asset and the mean variance efficient portfolio of risky assets. Thus all investors should hold the same portfolio of risky assets. Here, the risky assets are bonds and stocks, and the recommendations imply very different ratios of bonds to stocks. Thus the risky asset holdings of the three types of investors will have very different portfolio shares for stocks and bonds (as seen in the next table provided in the question).

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(b) Work out the expected returns and standard deviations of the risky asset portfolios recommended for the three investor types (conservative (C), moderate (M), aggressive (A)). Call these three portfolios p^C, p^M, and p^A.

The expected returns of the portfolios are weighted averages of the expected return for bonds and the expected return for stocks

$$E(r) = x_b E(r_b) + x_s E(r_s)$$

Thus

$E\left(r_{p^C}\right) = 0.5 \times 0.07 + 0.5 \times 0.12 = 0.0950, 9.50\%$

$E\left(r_{p^M}\right) = 0.3333 \times 0.07 + 0.6667 \times 0.12 = 0.1033, 10.33\%$

$E\left(r_{p^A}\right) = 0.2 \times 0.07 + 0.8 \times 0.12 = 0.1100, 11.00\%$

The variances of the portfolios are

$$\sigma^2 = x_b^2 \sigma_b^2 + x_s^2 \sigma_s^2 + 2 x_b x_s \rho_{b,s} \sigma_b \sigma_s$$

The standard deviations are the square roots of the variances. Thus

$\sigma_{p^C}^2 = 0.5^2 \times 0.09^2 + 0.5^2 \times 0.2^2 + 2 \times 0.5 \times 0.5 \times 0.25 \times 0.09 \times 0.2 = 0.014275$

$\sigma_{p^C} = \sqrt{0.014275} = 0.1195$

$\sigma_{p^M}^2 = 0.3333^2 \times 0.09^2 + 0.6667^2 \times 0.2^2 + 2 \times 0.3333 \times 0.6667 \times 0.25 \times 0.09 \times 0.2$
$= 0.02068$

$\sigma_{p^M} = 0.1438$

$\sigma_{p^A}^2 = 0.2^2 \times 0.09^2 + 0.8^2 \times 0.2^2 + 2 \times 0.2 \times 0.8 \times 0.25 \times 0.09 \times 0.2 = 0.027364$

$\sigma_{p^A} = 0.1654$

It turns out that neither of p^C, p^M, and p^A are the mean-variance efficient portfolio of risky bonds and stocks. The mean-variance efficient portfolio of risky assets, p^{MVE}, consists of investing 62% in risky bonds, and 38% in stocks (take my word for it, don't try to work it out). It has an expected return of 8.90% and a standard deviation of 10.49%.

Let us construct some better portfolio advice. For simplicity, focus on the aggressive investor.

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The efficient portfolio q lies on the optimal capital allocation line. Portfolios on this line are combinations of the riskless asset and the mean variance efficient portfolio.

$$\sigma_p = x_{MVE}\sigma_{MVE} \quad \Longleftrightarrow$$
$$0.1654 = x_{MVE} \times 0.1049$$

The portfolio shares in portfolio q are thus

$$x_{MVE} = 1.5765$$
$$x_f = 1 - x_{MVE} = -0.5765$$

and the expected return of portfolio q is

$$\begin{aligned} E(r_q) &= x_f E(r_f) + x_{MVE} E(r_{MVE}) \\ &= -0.5765 \times 0.04 + 1.5765 \times 0.0890 = 0.1172 \end{aligned}$$